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Preservice Teachers' Mapping Structures Acting on Representational Quantities[1]

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Abstract

In this article, I write about my research on five preservice secondary teachers' (PST) understanding and sense making of representational quantities associated with magnetic color cubes and tiles. Data came from individual interviews during which I asked PST problems guided by five main tasks: prime and composite numbers, summation of counting numbers, odd numbers, even numbers, and polynomial expressions in x and y. My work drew upon an analysis framework (Behr et. al, 1994) supported by a unit coordination construct (Steffe, 1988) associated with linear and areal quantities inherent in the nature of figures produced by these PST. Linear quantities can be thought of as generated via linear measurement units (e.g., inches, centimeters, units) whereas areal quantities are generated via areal measurement units (e.g., square inches, square centimeters, square units, etc.) I used thematic analysis supported by constant comparison and retrospective analysis to explain my theories and hypotheses concerning PST's representational quantities. I developed a data analysis framework which I named "Relational Notation" to describe these PST's understanding of linear and areal units. PST also treated the quantitative multiplication and addition operations as some kind of functions, mappings, when expressing the area of their growing rectangles made of magnetic color cubes and tiles as sums and products. Their behavior necessitated the existence of another component for my data analysis framework which I called "Mapping Structures"

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Preservice Teachers' Mapping Structures Acting on Representational Quantities

Mathematics gives rise to quantities which can be represented with physical objects mostly referred to as manipulatives. When representing these quantities, both students and their teachers should be proficient in identifying the characteristics of those quantities. Whole numbers can be thought of as made of units of 1. For instance, the number 12 can be thought of as a collection of twelve singleton units, or as four 3-units, or even as six 2-units. Moreover, in a representational situation, twelve little black square tiles can be used to model the representational quantity *12.* Odd integers can be represented as symmetric L-shaped figures, while even integers can be represented as rectangles with dimensions 2 by half the integer made of color tiles (Caglayan, 2006). Prime and composite numbers can be represented with various rectangular configurations made of color tiles as well (Caglayan, 2007). Color tiles serve to model not only positive integers, but also polynomial expressions in x and y.

Representation of irreducible quantities such as color tiles representing a "1", an "x", and a "y"; as well as larger representational quantities made of these sub-quantities is reminiscent of the "unitizing process." All the little pieces (e.g., each magnetic color cube denoting a "1" of a special number, each different size tile piece denoting a "1", an "x", a "y", an "x^2", an "xy", or a "y^2") and their various combinations (e.g., a 4 by 2 rectangle – made of 8 irreducible units of 1 – conceptualized as the unitizing of the even number 8, a $2x+y+1$ by $x+3y+2$ rectangle – made of 2 irreducible units of x^2, 3 irreducible units of y^2, 7 irreducible unit of xy, 5 irreducible units of x, 2 irreducible units of 1, 5 irreducible units of y – conceptualized as the unitizing of the polynomial expression $2x^2+7xy+3y^2+5x+5y+2$) serve for an essential theoretical construct which I define as Representational Unit Coordination.

Theoretical Framework

In its true nature, coordination is about "making various different things work effectively as a whole[2]." In the context of my study, it refers to the conception of unit structures in relation to smaller embedded sub-units within these unit structures, or, bigger units formed via iteration of these unit structures. In the multiplicative situation, for instance, the conception of 5 as 5 units of 1 is one way of coordinating units: 5 as a (composite) unit of 1. As another example, 35 can be

[2] Cambridge Advanced Learner's Dictionary online. Retrieved December 31, 2006 from http://dictionary.cambridge.org/define.asp?key=17018&dict=CALD

coordinated multiplicatively as 5 (composite) units of 7 (composite) units of 1. Unit coordination has been previously studied by various researchers in the mathematics education field. For example, Steffe analyzed the coordination of different levels of units in whole number multiplication problems which is reminiscent of a key concept in multiplication, i.e., the notion of composite units (Steffe, 1988). The essence of multiplication lies in fact in distributive rather than repeated additive aspect (Confrey & Lachance, 2000; Steffe, 1992). In the example above, the multiplication of 5 by 7 can be thought of as the injection of units of 7 (each being units of 1) into the 5 slots of 5, each slot representing a 1. In this example, the conceptualization of each singleton unit describing a unity, i.e., 1, stands for a *first level of unit coordination*. Moreover, 5 and 7 can be conceptualized (as composite units of 1) as 5×1 and 7×1, respectively1, as a *second level of unit coordination*. The product 5×7 which denotes 5 (composite) units of 7 (composite) units of 1, can be conceptualized as a *third level of unit coordination*. Some other researchers also studied unit coordination in a fractional situation (e.g. Lamon, 1994; Olive, 1999; Olive & Steffe, 2002). Additionally, work on intensive (e.g., miles per hour) and extensive quantities (e.g., number of hours) reflect unit coordination as well (Kaput, Schwartz, & Poholsky, 1985; Schwartz, 1988; Author (Olive & Caglayan, 2007) work on quantitative unit coordination and conservation also takes the unit coordination issue into account.

Representational Unit Coordination can be defined as the different ways of categorizing units arising from the modeling of identities on representational quantities as area as a product and area as a sum of the corresponding special rectangles made of magnetic color cubes and tiles. In my study, preservice teachers started with the area as a product concept. In its most basic sense, e.g., area of a rectangle, is defined as the product of its two dimensions. I am talking about the area of a rectangle, and not any other geometric figure, because the identities of representational quantities students analyzed via magnetic color cubes or tiles were always about a rectangle – prime rectangle, composite rectangle, odd rectangle, even rectangle, addition of odd and even integers generated as a growing rectangle, and polynomial rectangle. Coordination of these two dimensions, i.e., the arrangement of these two linear units in a particular order as an ordered pair such as (a, b) or (b, a), defines the first part of my construct RUC: *Multiplicative Type RUC*. The analysis of the other important concept, area as a sum (of a special number rectangle), is prone to several, not necessarily hierarchical levels of RUC. *Additive Type RUC* stands for the coordination, the arrangement of (in general two or more) areal units as n-tuples such as [2, 2, 2] or [3, 3] for the composite rectangle of 6. For this RUC type, areal units being coordinated have something in common. For instance, for the composite rectangle of 6, the 2's in [2, 2, 2] are interesting because 2 is a factor of 6, which is why this special additive type RUC is called *Equal Addends Type RUC*. Moreover, the coordination of less interesting (irreducible) areal units (of 1) as n-tuples such as [1, 1, 1, 1, 1, 1] for the same example, composite rectangle of the special number 6, necessitates the existence of another additive type RUC which is called *Irreducible Addends Type RUC.* There are actually ten additive type RUC's.

Context and Methodology

I conducted my study with PST's enrolled in the Mathematics Education Program of a university in the southeastern United States. I interviewed 5 PST's individually twice during January and February 2007. Duration of each session was about 60-75 minutes and each interview session was videotaped. The focus was on problems on identities for prime and composite numbers along with summation of counting numbers, odd and even integers as well as products and factors of polynomials modeled with magnetic color cubes and tiles. I selected my participants from two different undergraduate level mathematics education classes. Ben, Stacy and John came from the "Concepts in Secondary School Mathematics" class of 11 enrolled preservice teachers while Nicole and Ron came from the "Teaching Geometry and Measurement in the Middle School" class of 22 enrolled preservice teachers. All these five students volunteered to participate in my study. All names in this study are pseudonyms.

I used thematic analysis supported by constant comparison of the interviews and retrospective analysis. I also simplified and extended the *generalized notation for mathematics of a quantity* (Behr et al., 1994) in such a way as to cover identities that equate summation and product expressions of special numbers.

Results

In the context of prime and composite numbers, I asked the PST to represent a composite number (e.g., 15 and 28) and a prime number (e.g., 5 or 7) using wooden cubes. As for the summation of counting/odd/even numbers, the common direction for all the interview students was to represent counting/odd/even numbers using different magnetic color cubes for each number and add them so that they generate a growing rectangle. In the context of polynomials in x and y, they were asked to represent products and factors of polynomial expressions using different sized color tiles.

Multiplicative Representational Unit Coordination (MRUC)

Multiplicative type RUC arose from various usages by the PST such as "It [areal 6] is [linear]

6 and [linear] 1", "This [linear x] and this [linear 1] to find this [areal x]", "2 by one half of the number", "When you put this length [linear 1] and that length [linear 1] together", "This edge [linear 1] right here and this edge [linear 1] right here", "This edge [linear x] by this [linear x] edge". For all such usages, I used a relational notation of the form (a,b) where a and b stand for the corresponding linear quantities represented by the dimension tiles.

In the "Summation of Counting Numbers" activity, all PSTs came up with a similar growing rectangle pattern as in figure 1. When I invited Nicole to discuss the subunits corresponding to odd and even numbers on her growing rectangle, Nicole said that the odd numbers are all represented with straight lines. MRUC can not be concluded from her usage "straight lines" because a line is of linear nature and one can not have a relational notation of ordered pair representing a linear quantity. As for the even number subunits, however, she said that they all have length greater than 1, which lead me to think that she was beginning to think in a multiplicative way, as reflected in the following protocol.

Figure 1. Growing Rectangle Sequence Based on Rectangular Subunits.

Protocol 1: Nicole's Multiplicative RUC of Even Number Subunits.

I: What is common about the even numbers?

N: They are all by 2. Because in all even numbers 2 is a divisor or factor.

I: How would you describe the area of this big rectangle as a product?

N: I would say three times seven. Three inches times seven inches would give me 21 inches squared.

From this protocol, I deduce that the area of the growing rectangle, in the context of multiplicative RUC, can be expressed via the relational notation of ordered pair (3,7) of linear units, as described by Nicole's usage "three times seven". Her last usage "three inches times seven inches would give me 21 inches squared" calls for an operational type terminology "Mapping Structures" which will be described below. Nicole was able to see the multiplicative nature of the area of the growing rectangle for the special case corresponding to the 6^{th} stage (figure 1), however, she could not generalize this for any growing rectangle of the sequence. As for the subunits, she specified only one of the dimensions, as can be inferred from her usage "They are all by 2. Because in all even numbers 2 is a divisor or factor". This usage calls for a relational notation of the form $(2,\cdot)$ where the dot "·" represents the missing unspecified linear unit with value "half the even number". Ben was not able to generalize the linear unit corresponding to "any even number", either.

When I asked Ron about the pattern generating his growing rectangle for the addition of counting numbers activity, he said that every time you get an odd number, you can put it below the growing rectangle. The following protocol illustrates this point.

Protocol 2:

Ron's Description of the Growing Rectangle Sequence:

Bridge Connection between Consecutive Subunits.

R: The next even number will add two rows to what... see when we had three [meaning when he added the odd integer three] it was three by two [meaning the growing rectangle]. So you add two by two which is four, you get two more rows [meaning, two more cubes right next to the odd integer three] and it makes it [inaudible] to get a five. Now you have five, and for the next odd number you are gonna need a seven, which is why you add a 2 by 3 rows [meaning the two extra cubes will come from the even number 6].

Ron realizes that each even number subunit of the sequence serves as a bridge that connects the two consecutive odd number subunits. I can also infer that, just by looking at his growing rectangle sequence (similar to figure 1) that Ron knows that the difference between any two consecutive odd integers is 2. And "that 2" in fact is provided by the even integer subunit which is placed right in between the consecutive odd integers. Ron was the only student to make use of this strategy, which I name *Bridge Connection between Consecutive Subunits*. In Thompson's (1988) words, "To reason quantitatively is to reason about quantities, their magnitudes, and their relationships with other quantities" (p. 164) . Ron's *Bridge Connection between Consecutive Subunits* strategy has a strong indication of the quantitative reasoning described by Thompson. Ron's subunits are not only of multiplicative (and additive) nature – hence can be described via a multiplicative (and additive) type relational notation – but Ron is also very explicit in how these quantities exist on their own as well as in relation to their neighbors, which serve as a bridge at each step. And it is because of these bridges that all subunits and the growing rectangle made of those subunits come to exist as quantities for Ron.

I then asked Ron what is common about odd numbers and their area as a product because I wanted him to make a generalization for the odd numbers. He said that all the odd numbers can be represented by a rectangle whose dimensions are 1 by the odd number itself, at the same time pointing to their rectangles. Ron's generalization about odd number subunits could be expressed as a relational notation of ordered pair (1,n) of linear units where n represents the value of the corresponding odd number itself. Recall that Nicole described the odd integers simply as "straight lines", which was lacking a multiplicative nature. Ben also did not make a generalization about the dimensions of odd numbers. Ben and Nicole were alike in that they were successful in providing a multiplicative type RUC for special cases, though. Ron excelled in that he emphasized the multiplicative RUC for subunits standing for both odd and even numbers. In fact, when I asked him what is common about even numbers, he said that they are all split in two columns. His usage "2 by one half of the number" calls for a relational notation of ordered pair (2,n/2) of linear units where n stands for any even number of the sequence.

Additive Representational Unit Coordination (ARUC)

I observed more than one additive type RUC's which can be described using a functional notation $\sum f(i)=g(n)$ where areal quantities f(i)'s are being summed from 1 to n (number of addends) and i is the stage number (ordering number for the addends).

1) Equal Addends

   These are the addends describing a composite number rectangle. With the functional notation, f(i)=c, for all i. All PST produced this type.

2) Irreducible Addends (Type I)

   PST used these addends mostly when dealing with prime number rectangles. This is a special case for equal addends, f(i)=c, for all i, with c=1.

3) Symmetric Addends

   This type came from Ben's work on the summation of odd integers activity. Ben used these addends to describe the odd integers as symmetric L-shapes (Figure 2). For each symmetric L-shape, there are three addends only, i.e., n=3. One of these addends is equal to 1, and the remaining two addends are equal to each other. In other words, with the functional notation, one can write, f(2)=1, f(1)=f(3). For example, for the case of areal 9, which denotes an odd integer, f(2)=1, f(1)=f(3)=4. Note that f(1)+f(2)+f(3)=4+1+4=9, i.e., the odd integer itself.

Figure 2. Growing Square Sequence Based on L-Shape Subunits.

4) N+(N-1) type Addends

These are the addends describing a symmetric L-shape odd integer. In this case, n=2. With the functional notation, f(1)=N, and f(2)=N-1, i.e., the addends differ only by 1. John, Stacy, and Ben explained their ideas using this type when working on the summation of odd integers activity.

5) (N+1)+(N-1) type Addends

These are the addends describing a nonsymmetric L-shape even integer. Once again n=2. Only John referred to this type when working on the summation of even integers activity. A nonsymmetric L-shape even integer can be described using the functional notation f(1)=N+1, f(2)=N-1.

6) Recursive Addends

f(i+1) is being added to the previous summation (Nicole). With the functional notation, this can be written as g(n+1)=g(n)+f(i+1).

7) Summed Addends

The addends of the growing rectangle are areal units with different shapes made of magnetic color cubes representing the "area as a sum" part of the summation formula. For example, f(i)=i, f(i)=2i-1, f(i)=i+(i-1), f(i)=2i, for all i, for the addends corresponding to summation of counting numbers, odd numbers, odd numbers, and even numbers, respectively. 3 out of 5 PST (Nicole, Stacy, John) came up with this usage. Ron and Ben, on the other hand, did not care about the color shapes generating the growing rectangle. Instead, they used "Equal Addends" type RUC in expressing the area of the growing rectangle as a sum: Namely they treated the growing rectangle as a composite number rectangle.

8) Random Addends

n can be anything (Stacy). There are many different ways of writing the sum. With the functional notation, f(i)=anything, for all i. And f(i) is not necessarily equal to f(j) for any i≠j, where i, j denote the ordering number for the addends (areal units).

9) Irreducible Addends (Type II)

The area of the polynomial rectangle is written as the sum of irreducible areal units. 4 PST came up with this usage. For instance, for the 2x+y by x+2y+1 rectangle, the irreducible addends are [x^2, x^2, yx, xy, xy, y^2, xy, xy, y^2, x, x, y].

10) "Boxes of the Same Color" type Addends

The area of the polynomial rectangle is written as the sum of the boxes of the same color. As an example, only 1 PST used the areal units [$2x^2$, 4xy, 2x, yx, $2y^2$, y] to generate the 2x+y by x+2y+1 polynomial rectangle.

Mapping Structures (Ordinary and 2-Fold)

Stacy started by making long bars for the even numbers 2, 4, 6, 8, 10. She then added them to generate a growing rectangle sequence based on L-Shape subunits (Figure 3) similar to Ben, Nicole, and John.

Figure 3. Stacy's Growing Rectangle Sequence Based on L-Shapes.

She said that this pattern is similar to the odd integers. The protocol below illustrates this point.

Protocol 3: Stacy's Visual Proof Relating Two Summation Formulas.

S: The only difference is that we have an extra row [She splits the extra row as in Figure 4]

I: So you discovered the formula I guess...

S: Yeah... It would be n squared plus whatever that is [pointing to the extra row she just split]... n... n squared plus n yeah! [very excited]

Figure 4. Stacy's Decomposition of the Rectangle into a Square and a Bar.

As can be warranted by the protocol above, Stacy introduces n right after pointing to the extra row she just split. She therefore first visually locates both the growing square and the bar, and later on connects these objects to their dimensionalistic properties. Stacy's "extra row" formulation serves as a bridge between the two summation formulas. She just made the figure above and generalized it. She saw the square as an n by n square, and the bar as n (Resulting Addends). Her algebraic generalization was based on a representation (Figures 3 & 4), which was a special case for n=5. In Thompson's (1988) words, "To reason quantitatively is to reason about quantities, their magnitudes, and their relationships with other quantities" (page 164). Stacy was reasoning quantitatively by not only relating each L-shape even number subunit to the corresponding odd number component, but by connecting the two separate sequence of growing figures as well.

Stacy's discovery of n^2+n based on letters arising from the special case of n=5 made me excited because of her quick generalization in the conjectural process. I wanted to learn more from her about the meanings she would project onto these quantities, as illustrated in the protocol below.

Protocol 4: The Meanings Projected onto the *Resulting Addends* [n^2,n] and Mapping Structures.

I: Okay... n squared plus n... tell me more about that... What units have n squared and n?

S: Well... n square is n times n... so inch times inch it would be inches squared.

I: How about the extra n... is it an area or a length?

S: I don't know...

I: Is it in the area... that n?

S: Yeah... so it would be inches squared... by itself...

I: Does it make sense?

S: Yeah... well added together that has to equal an area so... since you are adding them together they have to have the same units... so it would be inches squared.

I: Okay... Where is the inches squared in n? In that n? [meaning the extra row]

S: It has to be in inches squared...

I: Okay... How do you say that? How do you figure?

S: It's just that it's n times 1... n inch and 1 inch... and then when you multiply them it'd be inches squared.

Stacy establishes arealness for her resulting addend n^2 by multiplying the corresponding same-valued linear quantities. These linear quantities are n inches and n inches. She demonstrates how the value-wise multiplication of n and n yields "value" n^2 and the "unit-wise" multiplication of inches and inches yields the measurement unit inches squared. When working on the first activity on prime and composite rectangles, Nicole also did the same thing. She multiplied the values of linear quantities as well as the measurement units attached to those quantities to produce a quantity of a totally different nature (Schwartz, 1988; Thompson, 1988). I infer a *2-fold Mapping Structures* for Stacy's representationally coordinated quantities. Both the ordered pair of values (n,n) and the measurement units (inches, inches) are mapped onto the corresponding value n^2 and the measurement unit in^2 through the multiplication operation behaving as a mapping. A *2-fold Mapping Structure* is slightly different from the *ordinary Mapping Structure* in that the fact that the multiplication mapping operates on both values and measurement units separately as in Stacy's usage "n squared is n times n... so inch times inch it would be inches squared". The ordinary Mapping Structures can be witnessed in Nicole and John's statements ("three inches times seven inches would give me 21 inches squared", "Length of 5 and width of 1 in which case the area would be 5") in their work with the prime and composite rectangles. Since the relational notation of ordered pairs already possesses the measurement units, both types of Mapping Structures have equivalent relational notations, such as (n,n) for the example above. A functional notation which describes both types of Mapping Structures can be written as $f(n,n) \rightarrow n^2$ or with the equality $f(n,n)=n^2$, where, f stands for the multiplication operation behaving as a mapping.

Stacy goes through a hesitation for a very short time in her sense making of the "extra row". Does it stand for a linear or an areal quantity? By her statement "since you are adding them together they have to have the same units... so it would be inches squared", I hypothesize that she establishes the arealness of this aforementioned quantity by deductive reasoning. The steps Stacy follows in her deductive reasoning can be outlined as follows:

1- The big n^2+n rectangle is an areal quantity.

2- The *resulting addend* n^2 is an areal quantity.

3- Therefore, the "other" *resulting addend n* must be an areal quantity, as well.

And finally, once establishing the arealness of the "extra row" quantity via deductive reasoning, Stacy then validates her judgment via inductive reasoning with reference to a *2-fold Mapping Structures*: She maps both the values and measurement units associated with the linear quantities into their areal counterparts as can be warranted by her statement "It's just that it's n times 1... n inch and 1 inch... and then when you multiply them it'd be inches squared." The 2-

foldness of the Mapping Structures arises in that both the ordered pair of values (n,1) and the ordered pair of measurement units (inches, inches) are mapped onto the corresponding value *n* and measurement unit in^2 through the multiplication operation behaving as a mapping. A functional notation which describes Stacy's verification can be written as $f(n,1)\rightarrow n$ or with the equality $f(n,1)=n$, where, f once again stands for the multiplication operation behaving as a mapping.

Notice that Stacy's behaviors in Protocol 4 above are always about the resulting addends generating the growing rectangle which can be modeled via a relational notation of ordered pairs $[n^2,n]$ of areal subunits. To be more specific, Stacy establishes the formation [(n,n),(n,1)]. The n^2, though was one of those big units (growing rectangles) in the context of the previous task on the summation of odd integers, behaves as a subunit in the context of the summation of even integers activity for Stacy. In fact, it is all about how Stacy wants to define this areal quantity by reference to her previous experience with the summation of odd integers activity. Remark that what Stacy establishes is the formation $[n^2,n]$ – equivalently, the formation [(n,n),(n,1)] – and not the formation (n,n+1) which denote equivalent quantities. In fact, when she was working with the "area of the growing rectangle as a product" column on the activity sheet, after providing the answers 4×5, 5×5, 6×7, 7×8, 8×9; I asked her whether these look like the expression n^2+n she discovered above. Stacy established the equivalence of these two formations as illustrated in the protocol below.

Protocol 5: Equivalence of $[n^2,n]$ and (n,n+1) Formations.

I: Does this look like n^2+n? [pointing to Stacy's written expressions on the "area of the growing rectangle as a product" column]

S: No. Does it? I don't think so...

I: Okay... Now I am gonna ask you to factor it...

S: Oh... n times n+1! [She writes $n^2+n=n(n+1)$ and very excited] It works...

I: Does it make sense?

S: Hm hm...

I: What units would you attach to n and n+1?

S: The n and the n+1 are both in inches.

I: What do you think about these as teaching tools?

S: It's cool though... I think it would work with summations. I did not learn the summations in high school though... I guess if you are teaching summations it should work...

Stacy's statements from Protocol 4 and Protocol 5 necessitate the existence of a theoretical construct which I name *Equivalence of Mapping Structures*. There must be an agreement of the ordered pair (n,n+1) of linear units and the ordered pair $[n^2,n]$ of areal units. These two formations can be reconciled via the equivalence of mapping structures. The multiplication operation, which behaves like a function, like a mapping, can be represented using a functional notation such as $f:(n,n+1)\rightarrow n^2+n$. Here, f denotes the multiplication operation which maps the linear units, n and n+1, which can be thought of as combinations of irreducible linear quantities (unit edges), onto the corresponding areal unit, namely n^2+n, which is also the same as the area of the growing rectangle itself. In other words, f acts on the ordered pair (n,n+1) of linear units and maps it onto the areal unit n^2+n. This operation can also be written as $f(n,n+1)=n^2+n$ with an "equals" sign.

Similarly, the addition operation behaves like a function, like a mapping, acting on irreducible areal quantities (unit blocks) or combinations of those. For instance, the function g, which represents the addition operation, acts on the ordered pair $[n^2,n]$ of areal units and maps it into the areal unit n^2+n. Using a functional notation, once again, this can be written as $g:[n^2,n]\rightarrow n^2+n$, or, with the equality $g[n^2,n]=n^2+n$. In other words, though they act on different types of representational quantities, the mappings f and g agree on one thing: That one thing is nothing but the fact that their image coincides (Figure 5). This is the essence of what is meant by "identity" in this research project. Area as a product coincides with the area as a sum eventually, thanks to these mapping structures.

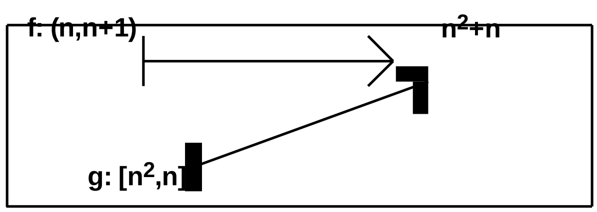

Figure 5, Equivalence of Mapping Structures.

John also made reference to *Mapping Structures* in his work with the summation of even integers. After generating a similar sequence of growing rectangles as in figure 6, he compared same-valued linear and areal quantities, as illustrated in the following protocol.

Protocol 6: John's Comparison of Linear and Areal Quantities and Mapping Structures.

I: How about the 6 and the 4 in the green... [L-shape representing the even number 10] are they areas or lengths?

J: Areas.

I: But this 6 is also the length of this rectangle [pointing to the 6 by 7 rectangle]. Are they [meaning the two 6] the same or different?

J: Different.

I: How are they different?

J: These 6 cubes by itself represent an area [pointing to the vertical part of the green L-shape figure] So this is... It's not just 6... It's 6 and 1.

Figure 6. John's Growing Rectangle Sequence Based on L-Shapes.

John's usage "It's not just 6... It's 6 and 1" can be explained using the *Mapping Structures* analysis model. The 2-foldness in these structures is missing as John does not mention unit-wise mapping (Compare with Stacy's behaviors indicating the 2-fold characteristic of the *Mapping Structures* in Protocol 4 above). John only maps the values of the linear quantities onto the value of an areal quantity. Multiplicative RUC arises from his usage "It's 6 and 1" however that's not the whole story. Multiplicative RUC is only a prerequisite for the construction of a *Mapping Structure*. In fact, John builds on the Multiplicative RUC by his statement "It's not just 6" which indicates the value of the areal quantity under consideration comes alive thanks to the multiplication operation which behaves as a mapping acting on the ordered pair (6,1) of linear units. For a *Mapping Structure* of multiplicative type to exist, therefore, one needs to establish the following conditions:

1- A pair ordering of the values of the linear quantities is mentioned.

2- The multiplication operation behaving as a mapping is acting on the ordered pair of these linear quantities.

3- The value of the areal quantity resulting from the mapping is indicated.

For a *2-fold Mapping Structure* to exist, on the other hand, the conditions above must hold as well as the following:

1'- A pair ordering of the measurement units of the linear quantities is mentioned.

2'- The multiplication operation behaving as a mapping is acting on the ordered pair of these linear measurement units.

3'- The measurement unit of the areal quantity resulting from the mapping is indicated.

In all the activities on magnetic color cubes, Nicole and John are the only students to make use of (ordinary) *Mapping Structures* as opposed to Stacy who excelled in the field with her reference to 2-fold Mapping Structures. Conditions necessitating the existence of ordinary and 2-fold *Mapping Structures* of additive type can be established in a similar manner as in 1, 2, 3, and 1', 2', 3' above.

4- An *n*-tuple ordering of the values of the areal quantities is mentioned.

5- The addition operation behaving as a mapping is acting on the ordered *n*-tuple of these areal quantities.

6- The value of the areal quantity resulting from the mapping is indicated.

4'- An *n*-tuple ordering of the measurement units of the areal quantities is mentioned.

5'- The addition operation behaving as a mapping is acting on the ordered *n*-tuple of these areal quantities.

6'- The measurement unit of the areal quantity resulting from the mapping is indicated.

The identification and the coordination of representational units of different types (multiplicative, additive) associated with color cubes and tiles are important aspects of quantitative reasoning and need to be emphasized during the teaching and learning process. Moreover, mapping structures serve as a bridge linking "area as a sum" and "area as a product" concepts in helping students and teachers make sense of identities on integers and polynomials: Why does the LHS have to be equal to the RHS?

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